Last time:

Integral of a function over a surface S

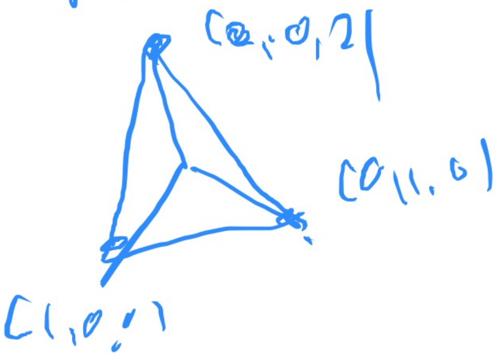
<u>Example</u> $\iint_S y \, dS$ — here: $f(x,y,z) = y$

where S was triangle with corners $(1,0,0)$, $(0,1,0)$ and $(0,0,2)$

$\Rightarrow$ parametrization via graph of function

triangle lies on plane $z = 2 - 2x - 2y$

param. via function $g(x,y) = 2 - 2x - 2y$



$0 \le z \le 2$

$\Rightarrow 0 \le 2 - 2x - 2y \le 2 \Rightarrow \underbrace{-2}_{-2} \le -2x - 2y \le 0$

$\Downarrow (-1)$

$2 \ge 2x + 2y \ge 0$

$\Rightarrow \boxed{0 \le x + y \le 1}$

Parametrizacia:

$$\Phi(u,v) = (u, v, g(u,v))$$

$$0 \leq u \leq 1$$
$$0 \leq v \leq u$$

$$\Rightarrow \|T_u \times T_v\| = \sqrt{\left(\frac{\partial g}{\partial u}\right)^2 + \left(\frac{\partial g}{\partial v}\right)^2 + 1}$$

$$g(u,v) = 2 - 2u - 2v$$

$$\frac{\partial g}{\partial u} = -2 \qquad \frac{\partial g}{\partial v} = -2$$

$\Rightarrow$ last class: $$\boxed{\begin{aligned} T_u \times T_v &= \left(-\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1\right) \\ &= (+2, 2, 1) \end{aligned}}$$

$$\iint_S y \, dS = \int_0^1 \int_0^u v \cdot 3 \, dv \, du = \int_0^1 \frac{3v^2}{2} \Big|_0^{1-u} du =$$

$$\Phi(u,v) = (u, v, 2-2u-2v)$$

$$\uparrow$$

$$y$$

$$\|T_u \times T_v\| = \|(+2, +2, 1)\| = \sqrt{(+2)^2 + (+2)^2 + 1} = 3$$

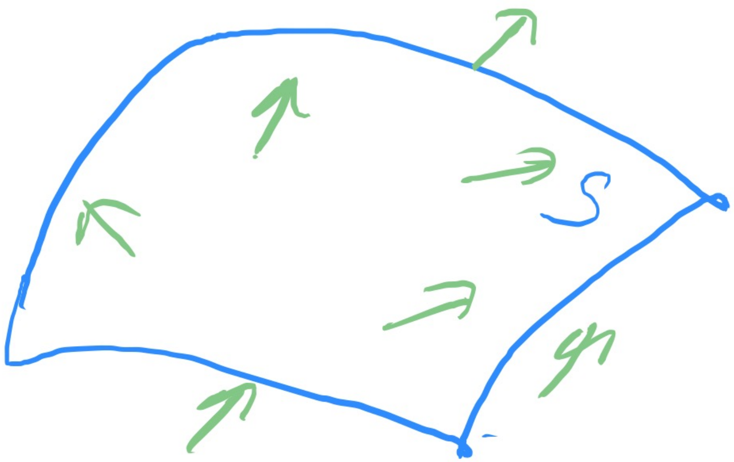
$$= \int_0^1 \frac{3(1-u)^2}{2} du = \ldots \text{ exercise } = \boxed{1/2}$$

<u>Important:</u> Surface integral of a function is independent of parametrization:

<u>Theorem:</u> If Φ_1 and Φ_2 are parametrizations of S
$\Rightarrow$ Value of integral $\int_S f \, dS$ same for both parametrizations.

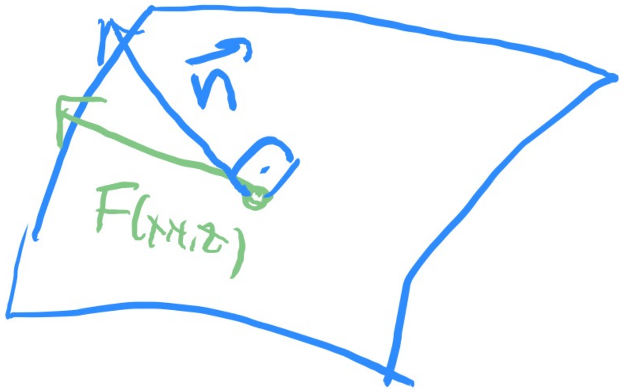
7.6 Surface Integrals for Vector Fields

<u>Motivation:</u>
- Given a surface S, say membrane
- a vector field $\vec{F}$ describing flow of a liquid



F vector field describing motion of a liquid (or gas) / time unit

<u>Question:</u> How much liquid goes through membrane / time unit?



$\vec{n}$ normal vector

relevant part of F

= part orthogonal to S

$\Rightarrow$ <u>Definition</u>: · S surface with parametrization $\Phi : D \to \mathbb{R}^3$

· F a vector field in $\mathbb{R}^3$

$\Rightarrow$

$$\boxed{\iint_{\Phi} F \cdot dS = \iint_{D} F(\Phi(u,v)) \cdot (T_u \times T_v)\, du\, dv}$$

Example:

Let S again be triangle with corners $(1,0,0)$, $(0,1,0)$, $(0,0,2)$

with parametrization $\Phi(u,v) = (u, v, 2-2u-2v)$,

$$\boxed{\begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le 1-u \end{array}}$$

Let $F(x,y,z) = (-y, x, z)$

Then

$$\iint_\Phi F \cdot dS = \iint_D F(\Phi(u,v)) \cdot (T_u \times T_v)\, dv\, du$$

$$= \iint_D F(u, v, 2-2u-2v) \cdot (2,2,1)\, dv\, du$$

$$= \int_0^1 \int_0^{1-u} (-v, u, 2-2u-2v) \cdot (2,2,1)\, dv\, du$$

$$= \int_0^1 \int_0^{1-u} -2v + \cancel{2u} + 2 - \cancel{2u} - 2v\, dv\, du$$

$$= \int_0^1 2v - 2v^2 \Big|_0^{1-u} du = \int_0^1 2(1-u) - 2(1-u)^2\, du = \boxed{1/3}$$

Question: How does integral change when we change parametrization?

Example: Take same triangle S as before,

$\Phi_2(u,v) = (v, u, 2-2u-2v)$

$0 \leq u \leq 1$

$0 \leq v \leq 1-u$

check:

- Φ_2 is also a parametrization of S
- $T_u = (0,1,-2)$, $T_v = (1,0,-2)$

  $T_u \times T_v = (-2,-2,-1)$

Doing same calculation as on previous page we obtain with same $F(x,y,z) = (-y, x, z)$

$$\iint_{\Phi_2} F \cdot dS = \int_0^1 \int_0^{1-u} (-v, u, 2-2u-2v) \cdot (-2,-2,-1) \, dv \, du$$

$$= \boxed{-1/3}$$

Observe: $\iint_{\Phi_2} F \cdot dS = -1/3 = \iint_{\Phi} F \cdot dS$

This is as bad as it can get!

Theorem Let Φ_1 and Φ_2 be parametrizations of the surface S

Let $\vec{n_1}$ and $\vec{n_2}$ be the normal vectors for Φ_1 and Φ_2

(recall: $\vec{n} = T_u \times T_v$)

let F be a vector field

$\Rightarrow$ (a) $\iint_{\Phi_1} F \cdot dS = \iint_{\Phi_2} F \cdot ds$ if $\vec{n_1}$ and $\vec{n_2}$ point to same side of S

(b) $\iint_{\Phi_1} F \cdot dS = -\iint_{\Phi_2} F \cdot ds$ if $\vec{n_1}$ and $\vec{n_2}$ point to opposite sides of S